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Frequency Response of Digitally Controlled Systems

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Multirate sampling techniques are used to sharpen and extend the concept of "frequency response" for discretely excited continuous systems. Frequency response, in this case, pertains to steady-state response of continuous variables at the input frequency and its positive aliases. Simple expressions are given for computing the amplitude ratios and phase angles as a function of these frequencies. Results are presented in terms of the familiar Bode plot. This Bode plot must be interpreted in a novel way, however. Similar expressions, appropriate for interpreting frequency response data obtained from sampled records of continuous responses, are also given. These frequency response methods apply for analysis of both open-loop and closed-loop discretely controlled continuous system responses, and are useful for assessing intersample ripple effects.

Nomenclature

$M(s)$	= data hold transfer function
s	= Laplace transform variable
T	= least common major sampling period
w	= bilinear transformation = $(\bar{z}-1)/(\bar{z}+1)$
w'	= $(2/T)w$
z	= $e^{sT/N}$
$\bar{z}$	= e^{sT}
ω	= angular frequency
ω_s	= angular sampling frequency = $2\pi/T$
$(\cdot)^T$	= impulse sampling of $(\cdot)$ at $1/$ Hz
$(\cdot)^{T/N}$	= impulse sampling of $(\cdot)$ at N/T Hz
$\mathrm{int}(\cdot)$	= integer part of $(\cdot)$

Introduction

WHEN a continuous, stable, stationary linear system is excited by a sine wave, the steady-state waveform is comprised of a single wave at the same frequency as the input. It differs from the input wave only by a phase angle and a magnitude factor. Moreover, it is unnecessary to compute the actual transient response of the system when the behavior for large values of time is of interest, since both the magnitude factor and phase angle can be read from a Bode plot.

A similar but more complex situation exists when a sampler and data hold couple the sine wave input to the continuous system. This configuration is referred to as a "discretely excited system." Given that the continuous system is stable, the continuous output waveform will contain a wave at the fundamental frequency and all of its aliases. Thus, if the system is forced with 1 sin bt, $0 \leq b < 2\pi/T$; the output will contain terms at frequencies b, $b+(2\pi/T)$, $b+(4\pi/T)$,... . The relative amplitudes and phase angles will depend on the data hold employed as well as the system transfer function. Nevertheless, given the data hold and system transfer functions, the magnitude and phase angle for each component can be read from a particular "Bode plot." This concept of frequency response is more comprehensive than the traditional concept of the "sampled spectrum" (see, e.g., Razazzini and Franklin[1] and Tou[2]), which is limited to determining the single sinusoid that fits the system output samples at the sampling instants.

In the sections that follow we review the frequency response concept for continuous systems and then proceed to the frequency response for discretely excited open-loop systems. The manner of application for single-rate sampled closed-loop systems is given next.

The paper is restricted to the analysis of single rate systems; its intent is the presentation of fundamental material necessary for an orderly transition to the multirate case. Thus the extensions relevant to the frequency response of multirate systems will be the topic of a future paper.

Continuous System Frequency Response

It will be helpful to first review the frequency response concept for continuous systems. Let $R(s)$ be a unit amplitude sine wave input to a system with a transfer function $G(s)$. The Laplace transform of the output, $C(s)$ is

$$C(s) = G(s)R(s) = G(s)\omega_0/(s^2+\omega_0^2) \tag{1}$$

Equation (1) can be expanded in partial fractions as

$$C(s) = \frac{A\omega_0}{s^2+\omega_0^2} + \frac{Bs}{s^2+\omega_0^2} + \begin{bmatrix} \text{terms associated with} \\ \text{characteristic poly-} \\ \text{nomial of } G(s) \end{bmatrix} \tag{2}$$

The bracketed term in Eq. (2) determines the characteristic behavior of the system which can be stable (negative eigenvalues) or unstable (positive eigenvalues). Nevertheless, the steady-state sinusoidal behavior is completely defined by the partial fraction coefficients A and B, since once they are known the steady-state time response can be written directly as

$$C(t)\Big|_{\lim\limits_{t\to\infty}} = A\sin\omega_0 t + B\cos\omega_0 t = \sqrt{A^2+B^2}\sin(\omega_0 t+\varphi) \tag{3}$$

where $\varphi = \tan^{-1}(B/A)$. The details of solving for A and B show clearly the relationship between the Bode plot and the

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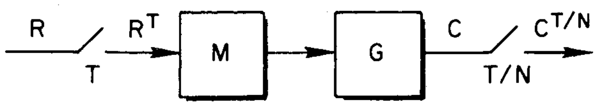

Fig. 1 Open-loop system.

steady-state waveform. To solve for A and B, multiply Eq. (2) by $[s^2+\omega_0^2]$ and evaluate the result for $s=j\omega_0$

$$G(s)\,\omega_0\,|_{s=j\omega_0}=(A\omega_0+Bs)\,|_{s+j\omega_0}+\begin{bmatrix}\text{terms associated with}\\ \text{characteristic poly-}\\ \text{nomial of } G(s)\end{bmatrix}(s^2+\omega_0^2)\,|_{s=j\omega_0} \tag{4}$$

or

$$G(s)\,|_{s=j\omega_0}=A+Bj=\sqrt{A^2+B^2}\exp[j\tan^{-1}(B/A)]=G(j\omega_0) \tag{5}$$

To summarize, we see that a sinusoidal input at frequency ω_0 produces a steady-state sinusoidal output waveform having the same frequency. It differs from the input only by a magnitude factor and a phase shift. Both magnitude factor and phase shift for any given input frequency ω_0 can be read directly from a Bode plot for $G(j\omega_0)$. That is, for any given input frequency ω_0,

$$A+jB=G(s)\,|_{s=j\omega_0} \tag{6}$$

The sections that follow expand this "frequency response" viewpoint to include discretely excited systems.

Open-Loop Frequency Response

Continuous Output Sampled

Consider the system of Fig. 1 where $G(s)$ represents an arbitrary transfer function and M represents an arbitrary data hold. Suppose R is a unit amplitude sine wave and the output is sampled with period T/N. Using multirate sampling results (see the Appendix),

$$C^{T/N}=[GMR^T]^{T/N}=(GM)^{T/N}R^T$$
$$=(GM)^{T/N}\frac{z^N\sin bT}{z^{2N}-2(\cos bT)z^N+1}\qquad (z\triangleq e^{sT/N}) \tag{7}$$

The notation follows Whitbeck and Hofmann.[3] The superscript denotes the period of the sampling operator. For example, R^T and $R^{T/N}$ denote sampling periods of T and T/N s, respectively.

The spectrum of Eq. (7) is of interest. It can be found by determining the spectrum of $C^{T/N}$ and taking the limit as $N\to\infty$. However, one must first obtain the N roots of the denominator in order to expand Eq. (7) in partial fractions. An essential identity, as derived by Whitbeck and Hofmann,[3] is given in Eq. (8)

$$z^{2N}-2(\cos bT)z^N+1$$
$$=\prod_{n=0}^{N-1}\left(z^2-2z\cos\left[\left(b+\frac{2\pi n}{T}\right)\frac{T}{N}\right]+1\right)$$
$$=\prod_{n=0}^{N-1}\left[\left(z-\cos\left[\left(b+\frac{2\pi n}{T}\right)\frac{T}{N}\right]\right)^2+\left(\sin\left[\left(b+\frac{2\pi n}{T}\right)\frac{T}{N}\right]\right)^2\right] \tag{8}$$

Using Eq. (8), the right-hand side of Eq. (7) can be expanded in partial fractions

$$C^{T/N}=\sum_{n=0}^{N-1}\frac{A_nz\sin\omega_n(T/N)+B_nz[z-\cos\omega_n(T/N)]}{z^2-2[\cos\omega_n(T/N)]z+1}+[\text{terms due to modes of }(GM)^{T/N}] \tag{9}$$

In Eq. (9)

$$\omega_n=b+2\pi n/T\qquad (n=0,1,2,\ldots,N-1) \tag{10}$$

For the present, we assume that $b\leq 2\pi/T$. Assume that responses in the modes of $(GM)^{T/N}$ approach zero as $t\to\infty$, i.e., that all modes are stable.

The steady-state waveform, *at the sampling instants,* can be written as

$$[C(t)]^{T/N}=\left[\sum_{n=0}^{N-1}(A_n\sin\omega_nt+B_n\cos\omega_nt)\right]^{T/N} \tag{11}$$

To solve for A_n and B_n, multiply each side of Eq. (7) by

$$[z^2-2[\cos\omega_k(T/N)]z+1]\qquad [0\leq k\leq(N-1)]$$

and evaluate for $z=1\measuredangle\omega_k(T/N)$. The only nonzero term on the right-hand side of the result occurs for $n=k$.

$$(GM)^{T/N}\frac{z^N\sin bT}{z^{2N}-2[\cos bT]z^N+1}\times[z^2-2[\cos\omega_k(T/N)]z+1]\,\Big|_{z=1\measuredangle\omega_k(T/N)}$$
$$=\sum_{n=0}^{N-1}\frac{[A_nz\sin\omega_n(T/N)]+B_nz[z-\cos\omega_n(T/N)]}{z^2-2[\cos\omega_n(T/N)]z+1}\times[z^2-2[\cos\omega_k(T/N)]z+1]\,\Big|_{z=1\measuredangle\omega_k(T/N)} \tag{12}$$

For any $n\neq k$, the right-hand side of Eq. (12) is identically zero since

$$z^2-2[\cos\omega_k(T/N)]z+1$$
$$=[z-\cos\omega_k(T/N)]^2+[\sin\omega_k(T/N)]^2 \tag{13}$$

vanishes when

$$z=1\measuredangle\omega_k(T/N)=\cos\omega_k(T/N)+j\sin\omega_k(T/N) \tag{14}$$

Evaluation of Eq. (12), subject to the constraint of Eq. (14), is a tedious chore.[3] However, the result is relatively simple

$$A_n+jB_n=\frac{1}{N}(GM)^{T/N}\Big|_{z=1\measuredangle\omega_n(T/N)}\qquad (z\triangleq e^{sT/N}) \tag{15}$$

Notice, in particular, the definition of z used in the evaluation.

To summarize, when the system is forced by $\sin bt$, the steady-state output waveform, sampled with period T/N, is given by Eq. (11). The coefficients A_n and B_n in Eq. (11) are computed using Eq. (15).

For example, let

$$M=(1-e^{-sT})/s\qquad G(s)=a/(s+a) \tag{16}$$

so that

$$\frac{1}{N}(GM)^{T/N}=\frac{1-e^{-aT/N}}{N(z-e^{-aT/N})}\,\frac{(1-z^{-N})}{(1-z^{-1})} \tag{17}$$

It is instructive to plot the amplitude ratio of the frequency response for Eq. (17) with N as a parameter. For the sake of clarity, the plot is versus ω rather than $\log\omega$.

Refer to Fig. 2, where $T=1$ s. Notice that over the range of frequency for the plot, $0\leq\omega\leq 8\pi/T$, the $N=1$ case amplitude ratio folds about the frequencies π/T, $3\pi/T$, ..., $7\pi/T$; the $N=2$ case folds about frequencies $2\pi/T$, $4\pi/T$, $6\pi/T$; and the $N=4$ case folds about $\omega=4\pi/T$. However, it can also be observed that the various amplitude ratio plots are periodic

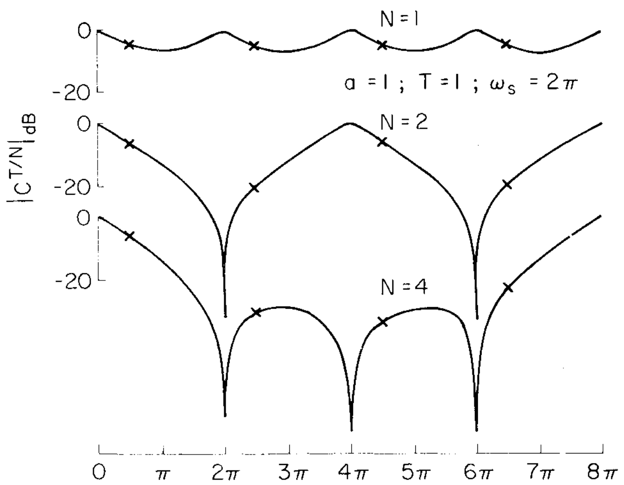

Fig. 2 Magnitude plot for $N = 1,2,4$.

with frequency. For $N=1$ this period is $2\pi/T$; for $N=2$, $4\pi/T$ and for $N=4$, $8\pi/T$. Each period contains precisely the number of frequency points (at the input or its positive alias frequencies) required to match the continuous steady-state time response at the sampling instants and at $N-1$ equally spaced intersample points. Consequently, *it is the sampling frequency interval rather than the folding frequency interval that is fundamental to generalizing the concept of frequency response.*

Consider the use of Fig. 2. Imagine that a unit amplitude sine wave with a frequency $b=\pi/2$ is input. In the $N=1$ case, our interest is matching the continuous steady-state time response only at the sampling instants with a single sine wave. The magnitude and phase angle (not shown in Fig. 2) can be read from this plot at $\omega=\pi/2$ (or $\pi/2+2\pi/T$, $\pi/2+4\pi/T$, $\pi/2+6\pi/T,...$; any of these points gives the correct value. Clearly, if the objective is to match at the sampling instants with a single sinusoid, the frequency could be b plus any $2\pi/T$ multiple. One cannot tell the difference once the waveform is sampled. In fact, the "sub"aliases at $b-2\pi/T$, $b-4\pi/T$ will also work. These subaliases are the difference terms prominent in modulation theory.

The $N=1$ plot in Fig. 2 corresponds to the "sampled spectrum" frequency response of sampled data control theory. Consider the $N=2$ case wherein the objective is to match one intersample point as well as the sample points. Let the input frequency, b, be $\pi/2$ and note that the points at $\omega=\pi/2$, $\pi/2+2\pi/T$ give the magnitude (and phase) for the sine waves at those frequencies, as would the points $\pi/2+4\pi/T$, $\pi/2+6\pi/T$. Suppose next that the input frequency is $b=\pi/2+2\pi/T$. Clearly, the second required component could be read from the "first alias" at $b+2\pi/T$ or the first subalias at $b-2\pi/T$ (or, for that matter, at a host of other frequencies).

In the $N=4$ case, four sine waves are required to fit three intersample points as well as the sample points. If the input frequency were $b=\pi/2+6\pi/T$, and if the plot of Fig. 2 with its limited range of 8π were the only one available, clearly it would be to our advantage to use the "difference" frequency points at $\omega=b-2\pi/T$, $b-4\pi/T$, and $b-6\pi/T$ to establish the magnitude and relative phase of the other three sine waves.

Consider generalizations of these observations. Notice the N sine wave components required to match the continuous steady-state time response at the sampling instants and at $N-1$ equally spaced intersample points need not have frequencies from within one period of the amplitude ratio plot. However, the frequencies used must each be an alias separated by $2\pi Nk/T$ (where k is an arbitrary integer for each frequency) from the N alias-related frequencies falling in any one period of the frequency response. This is evident in the

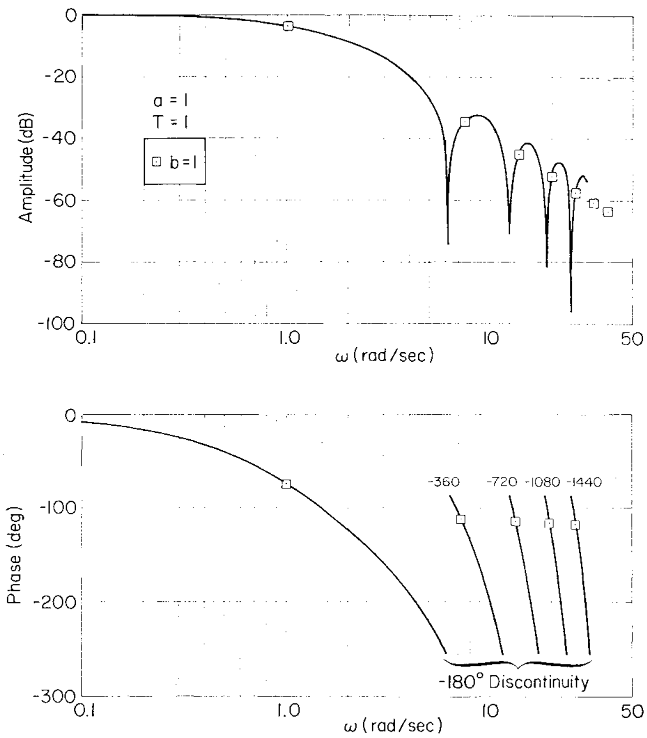

Fig. 3 Frequency response and spectral components of output.

example for $N=2$. If the input frequency is $\pi/2$, one would ordinarily choose sine wave components with frequencies $\pi/2$ and $(\pi/2+2\pi/T)$ to represent the continuous steady-state response at the sampling instants and one intersample point. However, alternative choices of frequencies are $\pi/2$ and $(\pi/2+6\pi/T)$ or $(\pi/2+2\pi/T)$ and $(\pi/2+4\pi/T)$, or $(\pi/2+4\pi/T)$ and $(\pi/2+6\pi/T)$, and so on. Choices of $\pi/2$ and $(\pi/2+4\pi/T)$ or $(\pi/2+2\pi/T)$ and $(\pi/2+6\pi/T)$ are *not* alternatives because these frequency choices do not satisfy the separation criterion stated previously.

This brief discussion associates aliases and subaliases with the sum and difference frequencies of modulation theory. It is not the case, however, that both sum and difference components are necessarily present simultaneously in the output. It was shown in the preceding that only N components are needed.

Continuous Output

In the previous section Eq. (15) was given. The continuous spectrum of the output variables of Fig. 1 can be determined by first finding the spectrum of $C^{T/N}$ and taking the limit as $N\to\infty$. As derived by Whitbeck and Hofmann,[3] the process of allowing N to approach infinity results in the identity of Eq. (18)

$$A_n+jB_n=\left.\frac{GM}{T}\right|_{s=j\omega_n} \tag{18}$$

ω_n is defined by Eq. (10) where now, however, $n=n_0$, $n_0+1,...$ and $n_0=-\text{int}(b/\omega_s)$.

The example of the previous section with N finite can now be studied for N infinite. This result gives the frequency response for the continuous output.

$$A_n+jB_n=\left.\frac{1-e^{-sT}}{sT}\cdot\frac{1}{s+1}\right|_{s=j\omega_n} \tag{19}$$

A Bode plot for this result is shown in Fig. 3. Components for the input frequency $b=1.0$ rad/s and its aliases have been indicated with the square symbols.

Interpretation of Fig. 3 is as follows. Suppose a unit sine wave at 1 rad/s is input to the sampler. Then, if the amplitude ratios and phase angles for sine waves with frequencies 1,

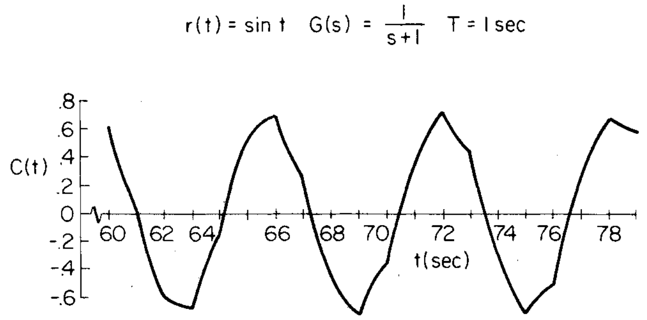

Fig. 4 "Steady-state" time response.

$(1+2\pi/T)$, $(1+4\pi/T)$, ..., are read from Fig. 3, and these components are added together, the resultant waveform will be an exact match of the actual steady-state output waveform in Fig. 4. One might expect this waveform to be very nearly sinusoidal, since the first alias is attenuated on the order of 30 dB, relative to the input component. However, the steady-state time response does not bear this out, as can be seen in Fig. 4. The reason is that the summation of alias terms is significant despite their small individual size.

Again, in connection with Figs. 3 and 4, consider a $(1+4\pi/T)$ rad/s unit sine wave input to the sampler. The steady-state output will contain subaliases at 1 rad/s and $(1+2\pi/T)$ rad/s; a component at the input frequency, $(1+4\pi/T)$ rad/s; and other aliases at $(1+6\pi/T)$, $(1+8\pi/T)$, ..., rad/s. The amplitude ratio and phase angle for each component is again read from Fig. 3 (at points indicated by the square symbols). When these components are added the steady-state time response is again that in Fig. 4. Thus, the continuous response spectrum and steady-state time response are the same for a sine wave input of given frequency as for a sine wave input having a positive frequency which is an alias of the given frequency.

Another observation is that the aliases do not represent "harmonic" components but rather represent modulation components that must add together properly in order to match conditions at the sampling instants for the basic period T. It can be seen in Fig. 4 that the "steady state" does not necessarily imply "periodic." Periodic steady-state time response waveforms occur only when the input frequency and the sampling frequency have an integer relationship with respect to one another.

Single-Rate Closed-Loop Frequency Response

Closed-loop results are dependent upon the configuration of the digital loops. However, the basic analysis procedure is independent of configuration. It is important to understand this procedure, and, in particular, the simplifications that occur in analysis of closed-loop systems.

Consider the (vector) system shown in Fig. 5. The objective is to find the coefficients that characterize the spectral components of the continuous response C, i.e., the frequency response. The procedure for this example is typical. First, solve for the vector component at the input of the data holds.

$$E^T = G_1^T R^T - G_1^T G_2^T (GM)^T E^T \tag{20}$$

Therefore,

$$E^T = [I + G_1^T G_2^T (GM)^T]^{-1} G_1^T R^T \tag{21}$$

Next, solve for $C(s)$

$$C = (GM)[I + G_1^T G_2^T (GM)^T]^{-1} G_1^T R^T \tag{22}$$

As in the open-loop case, the spectrum of $C(s)$ is of interest. It is found by first finding the spectrum of $C^{T/N}$ and then taking the limiting case of $N \to \infty$.

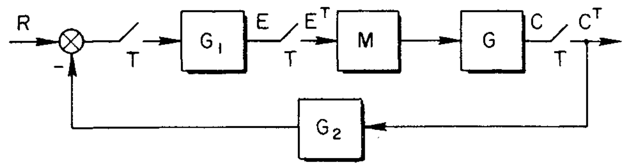

Fig. 5 Illustrative vector closed-loop configuration.

Le the input be a unit sine wave at frequency b rad/s and let

$$z \triangleq e^{sT/N} \tag{23}$$

so that

$$R^T = \frac{z^N \sin bT}{z^{2N} - 2(\cos bT) z^N + 1} \tag{24}$$

Using Eq. (22) and the results of the Appendix,

$$C^{T/N} = (GM)^{T/N} [I + G_1^T G_2^T (GM)^T]^{-1} G_1^T R^T \tag{25}$$

For the sake of brevity write Eq. (25) as

$$C^{T/N} = G_A^{T/N} G_B^T R^T \tag{26}$$

Expand the right-hand side of Eq. (26) in partial fractions

$$\begin{aligned} C^{T/N} &= G_A^{T/N} G_B^T \frac{z^N \sin bT}{z^{2N} - 2(\cos bT) z^N + 1} \\ &= \sum_{n=0}^{N-1} \frac{A_n z \sin\omega_n (T/N) + B_n z [z - \cos\omega_n (T/N)]}{z^2 - 2[\cos\omega_n (T/N)] z + 1} \\ &\quad + [\text{terms due to modes of } G_A^{T/N} G_B^T] \end{aligned} \tag{27}$$

Assume that responses in the modes of $G_A^{T/N} G_B^T$ approach zero as $t \to \infty$, i.e., that all closed-loop system modes are stable. In Eq. (27), ω_n is again defined by Eq. (10).

Notice that Eq. (27) has the same form as Eq. (7) if $(GM)^{T/N}$ is replaced by $G_A^{T/N} G_B^T$. Hence a crucial result is obtained using Eq. (15):

$$A_n + jB_n = \frac{1}{N} G_A^{T/N} G_B^T \Big|_{z = 1 \measuredangle \omega_n (T/N)} \tag{28}$$

But,

$$G_B^T(\bar{z}) \triangleq G_B^T(z^N) \qquad (\bar{z} \triangleq e^{sT}) \tag{29}$$

because of the local definition of z as $e^{sT/N}$. Therefore, using

$$[1 \measuredangle \omega_n (T/N)]^N = 1 \measuredangle \omega_n T = \cos\omega_n T + j\sin\omega_n T$$

$$= \cos[b + (2\pi n/T)]T + j\sin[b + (2\pi n/T)]T = \cos bT + j\sin bT$$

we obtain

$$G_B^T(z^N)\,\Big|_{1 \measuredangle \omega_n (T/N)} \equiv G_B^T(\bar{z})\,\Big|_{\bar{z} = 1 \measuredangle bT} \tag{30}$$

Clearly, z^N in G_B^T can be replaced with $\bar{z}$ provided it is evaluated for $\bar{z} = 1 \measuredangle bT$ instead of for $z = 1 \measuredangle \omega_n (T/N)$. This amounts to a simple (and very convenient) change of variable. At this point the result is

$$A_n + jB_n = \frac{1}{N} G_A^{T/N}(z) \Big|_{z = 1 \measuredangle \omega_n (T/N)} G_B^T(\bar{z}) \Big|_{\bar{z} = 1 \measuredangle bT}$$
$$(\bar{z} \triangleq e^{sT}, \quad z = e^{sT/N}) \tag{31}$$

Equation (31) is the basic result for the finite N case. To reiterate, to find the coefficients of the N sine waves matching the T/N sampled output C, compute the usual pulsed-transfer functions for

$$G_B^T(\bar{z}) = [I + G_1^T G_2^T (GM)^T]^{-1} G_1^T \qquad (\bar{z} \triangleq e^{sT}) \tag{32}$$

and evaluate for $\bar{z} = 1 \measuredangle bT$. Next, compute the usual T/N pulsed-transfer function for G_A as a function of z and evaluate it for $z = 1 \measuredangle \omega_n (T/N)$ where $\omega_n = b + (2\pi n/T)$.

It is superfluous to consider G_B^T as a function of z^N; it suffices to consider it as a function of $\bar{z}$. Moreover, only $G_A^{T/N}$ is a function of N; this simplifies obtaining the limiting case tremendously. For the case of $N \to \infty$, the continuous case, one obtains

$$A_n + jB_n = \left(\left[\frac{M(s)G(s)}{T} \right]_{s=j\omega_n} \right) \left([G_B^T(\bar{z})]_{\bar{z}=1 \measuredangle bT} \right) \qquad (\bar{z} \triangleq e^{sT}) \tag{33}$$

Equation (33) is the desired result for the given closed-loop configuration. However, it is the procedure that is key. One can follow the details through for other configurations quite easily.

The reader is cautioned that accurate numerical determination of $G_B^T|_{1 \measuredangle bT}$ may prove difficult at high sampling rates. This is the result of small differences between large numbers that occur in the computations as poles and zeros approach the unit circle. In this event, one is well advised to carry out equivalent computations in a domain where numerical conditioning is much improved (e.g., in terms of w' or w).

Corresponding results and illustrative examples for two-rate sampled closed-loop systems are given by Whitbeck and Hofmann.[3]

Conclusions

The "sampled spectrum" frequency-response concept of sampled data control theory is concerned with determining the simple sinusoid that fits the output of a single-rate system at the sampling instants. In this paper the frequency response concept has been extended to encompass the continuous spectrum for the continuous variables of a discretely controlled system. Moreover, the theory is sufficiently comprehensive to cover cases wherein a group of N sinusoids is used to match the continuous variables not only at the sample points but at $N-1$ equally spaced intersample points as well. N may be finite or infinite. Infinite N corresponds to the true continuous spectrum. The practical value of knowing the true continuous spectrum for continuous physical variables is self-evident. For example, this information can be used for evaluating intersample ripple and for guiding continuous filter selection. Cases wherein N is finite also have important practical application as, for example, in the bench validation of the digital controller hardware/software combination. Digital controller characteristics are often specified in terms of end-to-end "frequency response." On the bench, if continuous outputs (say from data holds) are sampled at a finite rate (N is finite), the difference between the measured frequency response and the true continuous frequency response ($N \to \infty$) may be significant. The results of this paper can be used to predict, minimize, or correct for the difference between measured and true frequency responses.

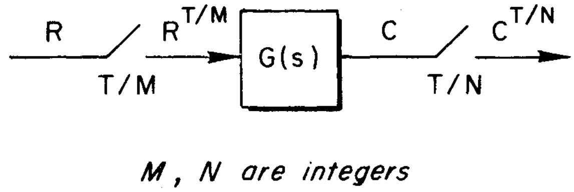

Fig. A1 A multirate sampled system.

Results for closed-loop cases depend upon the specific configuration of the digital loops. However, the basic analysis procedure is independent of the loop configuration. The procedure is to obtain the pulsed transfer function matrix relating the outputs to the data holds and the inputs through which the (sampled) sine wave inputs enter. The continuous system (including data holds) is placed in cascade with the output of the pulsed system. Frequency response is then evaluated in exactly the same manner as for any discretely excited open-loop continuous system.

Extensions for true multirate sampled closed-loop systems will be reported in a later paper. In addition, the theory is currently being extended to provide a tool for quantifying fidelity of digital simulations of continuous systems.

Appendix: Multirate Sampling Operator Notation

In the treatment of discretely controlled systems, impulsive sampling operations are assumed. Notation is made clear with the aid of Fig. A1. Superscripts are used to denote the period of each sampling operation. For example, the signal R is shown sampled with period T/M, while the output signal C is sampled with period T/N. Both M and N are positive integers. The output of Fig. A1 can be written as

$$C^{T/N} = [GR^{T/M}]^{T/N} = [G^{T/MN} R^{T/M}]^{T/N}$$

where inner and outer sampling operations do not commute in general. This intuitively obvious result has also been proved using either residue theory or vector switch decomposition theory.[3] Two well-known special cases of this result are

$M = 1$, N (a positive integer):[1]

$$C^{T/N} = [G^{T/N} R^T]^{T/N} \equiv G^{T/N} R^T$$

$N = 1$, M (a positive integer):[1]

$$C^T = [G^{T/M} R^{T/M}]^T$$

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